FMI NFA 2019-2020 - Homework 1

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Exercise. ([Dei85, exercise 5.1]) Let $f \in C(\mathbb{R}^n)$ be such that f maps $\partial B(0,r)$ onto itself, for some $r > 0$. Then

$$d(f^m, B(0,r), 0) = [d(f, B(0,r), 0)]^m.$$

Proof. We have $f(\partial B(0,r)) = \partial B(0,r)$, hence the set

$$\mathbb{R}^n \setminus f(\partial B(0,r)) = \mathbb{R}^n \setminus \partial B(0,r)$$

has only one bounded connected component, $B(0,r)$.

Inductively, by the product formula ([Dei85, theorem 5.1]),

$$\begin{aligned} d(f^m, B(0,r), 0) &= d(f^{m-1} \circ f, B(0,r), 0) = \\ &= d(f, B(0,r), B(0,r)) d(f^{m-1}, B(0,r), 0) = \\ &= \ldots = \\ &= [d(f, B(0,r), B(0,r))]^m = \\ &\overset{(d5)}{=} [d(f, B(0,r), 0)]^m. \end{aligned}$$

□

Exercise. ([Dei85, exercise 5.2]) If $\Omega \subseteq \mathbb{R}^n$ is open bounded and $f \in C(\overline{\Omega})$ is one-to-one, then $d(f, \Omega, y) \in \{-1, 1\}$ for every $y \in f(\Omega)$.

Proof. Fix $y_0 \in f(\Omega)$ and $x_0 = f^{-1}(y_0)$. Let $\{K_i\}_{i \in I}$ be the bounded connected components of $\mathbb{R}^n \setminus f(\partial\Omega)$. Denote by K_j the component that contains y_0.

By [Dei85, proposition 1.1], there exist continuous extensions $\widetilde{f}$ of f and $\widetilde{f^{-1}}$ of f^{-1} to $\mathbb{R}^n$.

By [Dei85, theorem 3.1(d6)], since $\mathrm{id} = f^{-1} \circ f$ and $\widetilde{f^{-1}} \circ \widetilde{f}$ coincide on the boundary of Ω, we have

$$d(f^{-1} \circ f, \Omega, x_0) = d(\widetilde{f^{-1}} \circ \widetilde{f}, \Omega, x_0).$$

Now the product formula ([Dei85, theorem 5.1]) implies that

$$\begin{aligned}
1 &\stackrel{d1}{=} d(\mathrm{id}, \Omega, x_0) = \\
&= d(f^{-1} \circ f, \Omega, x_0) = \\
&= d(\widetilde{f^{-1}} \circ \widetilde{f}, \Omega, x_0) = \\
&= \sum_{i \in I} d(\widetilde{f}, \Omega, K_i) d(\widetilde{f^{-1}}, K_i, x_0). \qquad (1)
\end{aligned}$$

We will now show that there is only one nonzero term, namely $i = j$. Fix $i \neq j$.
We first show that $\partial K_i \subseteq f(\partial\Omega)$. Since f is a homeomorphism, we have

$$f(\partial\Omega) = \partial f(\Omega). \qquad (2)$$

By definition of K_i, we have

$$K_i \cup \left(\bigcup_{\substack{m \in I \\ m \neq i}} K_m \right) \cup K_\infty = \left(\bigcup_{m \in I} K_m \right) \cup K_\infty = \mathbb{R}^n \setminus f(\partial\Omega) \stackrel{2}{=} \mathbb{R}^n \setminus \partial f(\Omega). \qquad (3)$$

Taking the boundaries of both sides, we obtain

$$\partial \left(K_i \cup \left(\bigcup_{\substack{m \in I \\ m \neq i}} K_m \right) \cup K_\infty \right) = \partial \left(\mathbb{R}^n \setminus \partial f(\Omega) \right). \qquad (4)$$

For the left side in (4), note that when A and B are disjoint open sets, we have $\partial(A \cup B) = \partial A \cup \partial B$. For the right side, note that the boundary of a set coincides with the boundary of its complement. Thus

$$\partial K_i \cup \partial \left(\bigcup_{\substack{m \in I \\ m \neq i}} K_m \right) \cup \partial K_\infty = \partial(\partial f(\Omega)) = \partial f(\Omega) \stackrel{2}{=} f(\partial\Omega),$$

which implies that

$$\partial K_i \subseteq f(\partial\Omega). \qquad (5)$$

In particular, $\partial K_i \subseteq f(\overline{\Omega})$, so $\widetilde{f^{-1}}$ and f^{-1} coincide on ∂K_i.
We can represent its closure as the disjoint union

$$\begin{aligned}
\overline{K}_i &= \partial K_i \cup K_i = \\
&= \partial K_i \cup [K_i \setminus f(\overline{\Omega})] \cup [K_i \cap f(\overline{\Omega})] = \\
&= \partial K_i \cup [K_i \setminus f(\overline{\Omega})] \cup [K_i \cap f(\Omega \cup \partial\Omega)] = \\
&= \partial K_i \cup [K_i \setminus f(\overline{\Omega})] \cup [K_i \cap f(\Omega)] \cup [K_i \cap f(\partial\Omega)].
\end{aligned}$$

By (5) we have that

$$\partial K_i \cup [K_i \cap f(\partial\Omega)] \subseteq f(\partial\Omega).$$

Because f is a homeomorphism and both Ω and K_i are open, $K_i \cap f(\Omega)$ and $K_i \setminus f(\overline{\Omega})$ are both open subsets of K_i. Since $x_0 \notin f(\partial\Omega)$, [Dei85, theorem 3.1(d2)] implies that

$$d(\widetilde{f^{-1}}, K_i, x_0) = d(\widetilde{f^{-1}}, K_i \setminus f(\overline{\Omega}), x_0) + d(\widetilde{f^{-1}}, K_i \cap f(\Omega), x_0).$$

The second term is zero because $y_0 \notin K_i \cap f(\Omega)$, i.e.

$$d(\widetilde{f^{-1}}, K_i \cap f(\Omega), x_0) \stackrel{(d6)}{=} d(f^{-1}, K_i \cap f(\Omega), x_0) \stackrel{(d4)}{=} 0.$$

Hence, for $i \neq j$, we have

$$d(\widetilde{f^{-1}}, K_i, x_0) = d(\widetilde{f^{-1}}, K_i \setminus f(\overline{\Omega}), x_0).$$

If we assume that

$$d(\widetilde{f^{-1}}, K_i \setminus f(\overline{\Omega}), x_0) \neq 0,$$

by [Dei85, theorem 3.1(d4)] there should exist $y \in K_i \setminus f(\overline{\Omega})$ such that $\widetilde{f^{-1}}(y) = x_0$. Thus

$$d(\widetilde{f}, \Omega, K_i) \stackrel{d5}{=} d(\widetilde{f}, \Omega, y) \stackrel{d4}{=} 0, \tag{6}$$

since $y \notin \widetilde{f}(\overline{\Omega}) = f(\overline{\Omega})$.

Hence for all $i \neq j$, either

$$d(\widetilde{f}, \Omega, K_i) = 0 \qquad \text{or} \qquad d(\widetilde{f^{-1}}, K_i, x_0) = 0,$$

so the sum in (1) reduces to

$$1 = \sum_{i \in I} d(\widetilde{f}, \Omega, K_i) d(\widetilde{f^{-1}}, K_i, x_0) = d(\widetilde{f}, \Omega, K_j) d(\widetilde{f^{-1}}, K_j, x_0). \tag{7}$$

Since $y_0 \in K_j$,

$$d(\widetilde{f}, \Omega, K_j) \stackrel{d5}{=} d(\widetilde{f}, \Omega, y_0) \stackrel{d6}{=} d(f, \Omega, y_0). \tag{8}$$

From (7) and (8) it follows that

$$d(f, \Omega, y_0) = d(\widetilde{f}, \Omega, K_j) = \frac{1}{d(\widetilde{f^{-1}}, K_j, x_0)}.$$

However, the topological degree d can only be an integer, hence

$$d(f, \Omega, y_0) = d(\widetilde{f^{-1}}, K_j, x_0) \in \{-1, 1\}.$$

□

References

[Dei85] Klaus Deimling. Nonlinear functional analysis. Springer-Verlag, 1985. ISBN: 0387139281.